RAN-2101001102040001

# RAN-2101001102040001

**M. A. (Part - II) External - Examination April - 2026**

**Mathematics**

**Advanced Functional Analysis (Paper - 504)**

**Time: 3 Hours ]** **[ Total Marks: 100**

| સૂચના : / Instructions | Seat No.: |
|---|---|
| (1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી.<br>Fill up strictly the above details on your answer book<br>(2) Answer all questions.<br>(3) Figures to the right indicates marks.<br>(4) Follow usual notations and conventions. | Student's Signature |

Q.1 (a) Let H be a Hilbert space and M is orthogonal set in H, then prove that M is total if and only if for all non-zero $x \in H$ Parseval's relation $\sum_k |\langle x, e_k \rangle|^2 = \|x\|^2$ holds. 07

(b) Prove that every bounded linear function $f$ on a Hilbert space H can be uniquely represented in term of inner product, i.e., $f(x) = \langle x, z \rangle$, where z depends on $f$ is uniquely determine by $f$ and has $\|f\| = \|z\|$. 07

(c) Let $H_1$ *and* $H_2$ be two Hilbert spaces and $S: H_1 \to H_2$ *and* $T: H_1 \to H_2$ be bounded linear operators ans $\alpha$ be any scalar, then prove that 06

(i) $\langle T^* y, x \rangle = \langle y, Tx \rangle$

(ii) $(S + T)^* = S^* + T^*$

(iii) $(\alpha T)^* = \bar{\alpha}\, T^*$

**OR**

Q.1 (a) Let $H_1$ *and* $H_2$ be two Hilbert spaces and $T: H_1 \to H_2$ be a bounded linear operator. Then prove that the Hilbert adjoint operator $T^*$ *of* $T$, an operator from $H_2$ *to* $H_1$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle, \forall x \in H_1, y \in H_2$, exists, is unique and is a bounded linear operator with norm $\|T^*\| = \|T\|$. 07

(b) Let $(T_n)$ be a sequence of bounded self- adjoint linear operators on a Hilbert space. Suppose sequence $(T_n)$ converges, i.e., $\|T_n - \mathrm{T}\| \to 0$ as $n \to \infty$ where $\|\,.\,\|$ is norm on $B(H, H)$ then prove that limit operator T is also self-adjoint linear operator on Hilbert space H. 07

(c) Prove that A linear operator T on a Hilbert space H is unitary iff Hilbert adjoint of T exist and $T^*T = TT^* = I$. 06

Q.2 (a) Prove that Two Hilbert spaces H and $\widetilde{H}$ either real or complex are isomorphic iff they have same Hilbert dimension. 07

(b) Let X be a real vector space and P be a sublinear functional on X. Let $f$ be linear functional defined on a subspace Z of X satisfying 07

$f(x) \leq P(x), \forall x \in Z$ then prove that $f$ has a linear extension $\hat{f}: Z \to X$ satisfying $\hat{f}(x) \leq P(x), \forall x \in X \text{ and } \hat{f}(x) = f(x), \forall x \in Z.$ 06

(c) Write Zorn's lemma. Prove that Every vector space $X \neq \{0\}$ has Hammel basis.

**OR**

Q.2 (a) Let $f$ be linear functional defined on a subspace Z of a normed space X then prove that there exists a bounded linear functional $\hat{f}$ on X, an extension of $f$ from Z to X with norm $\|\hat{f}\|_X = \|f\|_Z$ , where $\|\hat{f}\|_X = Sup_{\substack{x \in X \\ \|x\|=1}} |\hat{f}(x)|$ and $\|f\|_Z = Sup_{\substack{x \in Z \\ \|x\|=1}} |f(x)|$. 07

(b) Prove that For every fixed $x \in X$ ($where\ X\ is\ a\ normed\ space$) the functional $g_x$ defined by $g_x(f) = f(x)$ is bounded linear functional on $X'$ $and$ $\|g_x\| = \|x\|$. 07

(c) Let $U: H \to H$ $and$ $V: H \to H$ be unitary operators on a Hilbert space H, then prove that 06

(i) U is isometric and $\|Ux\| = \|x\|$

(ii) $\|U\| = 1$ $where\ H \neq \{0\}$

(iii) $U^{-1}$ is unitary

(iv) $UV$ is unitary

Q.3 (a) If a metric space $X \neq \emptyset$ be a complete, then prove that it is nonmeager in itself. Hence if $X \neq \emptyset$ is complete and $X = \bigcup_{k=1}^{\infty} A_k$ then at least one $A_k$ contains a nonempty open subset. 07

(b) Show that the converse of $UOC \Rightarrow SOC \Rightarrow WOC$ is not true in general, where UOC= Uniformly operator convergence, SOC = Strongly operator convergence, WOC= weakly operator convergence. 07

(c) Let X and Y be two normed spaces and $B(X, Y)$ , the set of all bounded linear operators X into Y then show that $(S + T) \in B(X, Y)$, $(S + T)^{\times} = S^{\times} + T^{\times}$ and $(\propto T)^{\times} = \propto T^{\times}$. 06

**OR**

Q.3 (a) If the dual space $X'$ of a normed space X is separable, then prove that X itself is separable. 07

(b) Prove that a bounded linear operator $T: X \longrightarrow Y$, where X and Y are Banach spaces, is an open mapping. Hence, T is bijective, $T^{-1}$ is continuous and thus bounded. 07

(c) Prove that in a normed space X, $(x_n) \xrightarrow{w} x$ iff 06

(i) Sequence $\|(x_n)\|$ is bounded

(ii) For every element of a total subset $M \subset X'$ we have $f(x_n) \to f(x)$

Q.4 (a) Let $T_n$ be a sequence of compact linear operators from a normed space X into a Banach space Y. If $T_n$ is strongly operator convergent then prove that T is compact. 07

(b) Let $T: H \to H$ be a self-adjoint linear operator on a complex Hilbert space H then prove that $\exists$ a number $\lambda \epsilon \rho(T)$ of T iff $\exists$ c $> 0$ such that for every x $\in H, \|T_\lambda(X)\| \geq c. \|X\|$. 07

(c) Prove that a bounded linear operator $P: H \to H$ on a Hilbert space H is a projection if and only if P is self-adjoint and idempotent. 06

**OR**

Q.4 (a) Define Closable Operator and minimal extension. Also let $T : D(T) \to H$ be densely defined linear operator in H, if T is self- adjoint then prove that T is symmetric but converse is not true. 07

(b) Prove that the spectrum $\sigma(T)$ of a bonded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space $H$ is real. 07

(c) Prove compactness of $T: l^2 \to l^2$ defined by $y = (\eta_j) = T_x$, where $\eta_j = \frac{\xi_j}{j}$ for j=1, 2, ... 06

Q.5 (a) If $U: H \to H$ is a unitary linear operator on a complex Hilbert space $H \neq \{0\}$, then prove that the spectrum $\sigma(U)$ is a closed subset of the unit circle; thus $|\lambda| = 1$ for every $\lambda \epsilon \sigma(U)$. 07

(b) Prove that The Cayley transform $U = (T - iI)(T + iI)^{-1}$ of a self-adjoint operator $T : D(T) \to H$ exists on H and is a unitary operator; here, $H \neq \{0\}$ is a complex Hilbert space. 07

(c) Let $T: \mathcal{D}(T) \to H$ be a linear operator, where $H$ is a complex Hilbert space and $\mathcal{D}(T)$ is dense in $H$. Then prove that if T is symmetric, its closure $\bar{T}$ exists and is unique. 06

**OR**

Q.5 (a) Let $T: X \to X$ be a compact linear operator from a normed space X. Then prove that for every $\lambda \neq 0$ the null space $\mathcal{N}(T_\lambda)$ of $T_\lambda = T - \lambda I$ is finite dimensional 07

(b) State and prove Weak Convergence theorem. 07

(c) Let $T : D(T) \to H$ be densely defined linear operator in H and suppose that T is injective and its range R(T) is dense in H, then prove that $T^*$ is injective and $(T^*)^{-1} = (T^{-1})^*$. 06

---